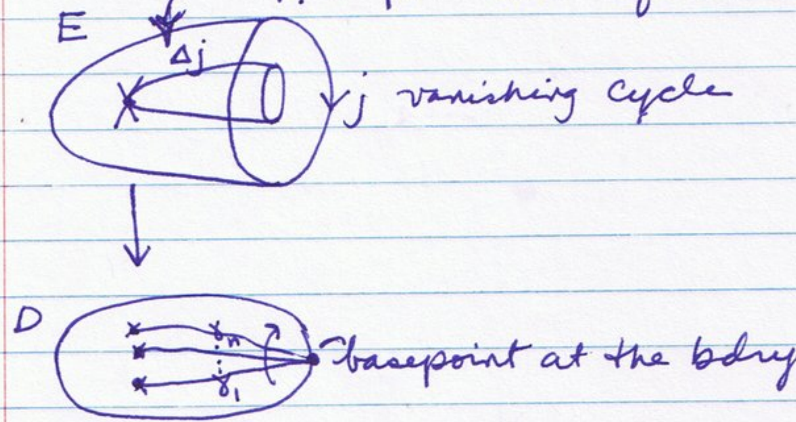
Let $\begin{matrix} E \\ \downarrow \pi \\ D^2 \end{matrix}$ be a Lefschetz fibration
exact, with usual assumptions

the thimble that lives above γ_j we call Δ_j.
M = fiber above reference point



In the picture we have fixed a reference fiber M in E, we've fixed a basis of vanishing paths $\gamma_1, \ldots, \gamma_n$ which join the basepoint to a critical value.

For each critical value we have a path & along that path we have a thimble. [This is a useful way to build Lags submflds in the fiber & the total space.]

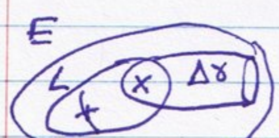
Today: Define $\underbrace{\mathcal{F}(\pi)}$ and see how it relates to $\mathcal{F}(E)$ & $\mathcal{F}(M)$.
Fukaya Category of a Lefschetz Fibration
a.k.a. "Fukaya-Seidel Category"

The objects in this category should consist of lagrangians in the total space & we want to include also a few select noncompact Lags. including thimbles.

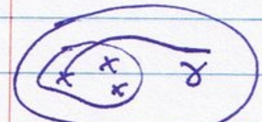
Idea: $\mathcal{F}(\pi)$: Objects are compact, closed exact Lag. submflds of E (i.e. the objects of $\mathcal{F}(E)$)
(+ graded spin submfld with whole package needed to do Floer theory).
- allow thimbles of π ($\Delta_1, \ldots, \Delta_m$ or any other thimbles for any vanishing path).

How do we define Floer theory for the whole thing?

- For intersecting closed exact Lagrangians, this we've discussed in previous lectures, we're in $\mathcal{F}(E)$.
- If dealing with closed exact Lagrangian & thimble of π, we're also in good shape:



All the intersection points will live in the interior. By Max Principle argument, any holomorphic disc will also stay in the interior so we don't need to be concerned about noncompactness of the thimble $\Delta\gamma$.



In summary:

operations on
$\cdot CF(L, L')$ OK ($\mathcal{F}(E)$)
$\cdot CF(L, \Delta)$ OK. by max principle
(Intersection $\xmapsto{\pi}$ interior(disc) & so do discs

- If I have two thimbles and I want to define their Floer theory, $\boxed{\text{what is } CF(\Delta_0, \Delta_1)?}$

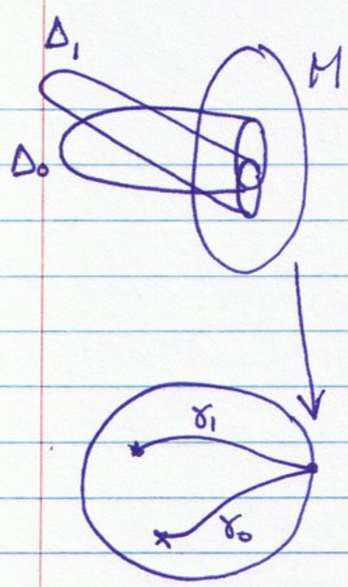
The difficulty is, ~~since~~ thimbles are Lagrangian submflds with bdry & they tend to intersect mostly on the bdry fiber. But we can push things a little to get the intersection off the bdry (since Floer theory should be invariant under isotopies & small deformations). However, when we push things a little, 2 things can happen

- push intersection into interior
- or push it outside past the boundary.

<u>So we need a rule for how to do this.</u>

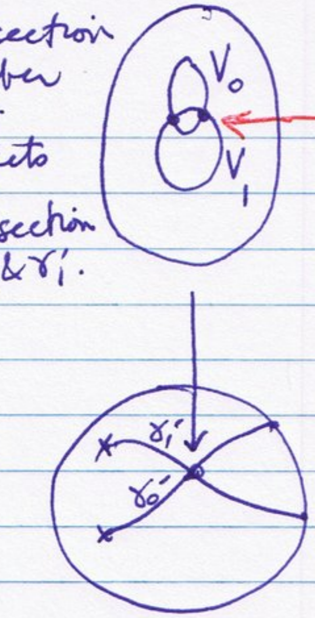
<u>Naive Rule</u>: Push endpoint of γ_0 in the + direction.
counterclkwise

CASE I:



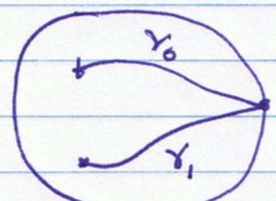
$\longrightarrow$ pushing up γ_0, counterclkwise, so endpoint further along bdry

Cross-section of fiber that projects to intersection of γ_0' & γ_1'.

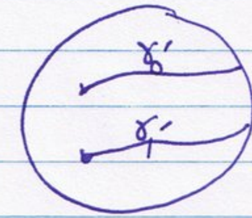


intersection points are in the interior of the fiber

CASE II:



$\longrightarrow$ pushing up γ_0, counterclkwise



In this case we declare that there are <u>no</u> intersections.

<u>Rule</u> (refined): Count intersection $V_0 \cap V_1$ in the reference fiber M only if γ_1 starts clockwise from γ_0.

[Now apparent why important that reference point at bdry, we know what it means to be clkwise from each other there.]

- <u>directed $^{(A_\infty-)}$ category of a basis of thimbles</u>

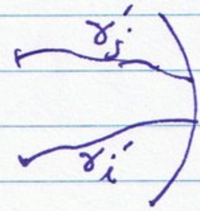
Given $(\gamma_1, \ldots, \gamma_m) \xrightarrow{\text{we had}}$ thimbles $(\Delta_1, \ldots, \Delta_m)$.

We <u>define</u>, (artificially) $\mathcal{F}^{\to}(\{\gamma_i\})$ has objects $\Delta_1, \ldots, \Delta_m$.

"$\to$" indicates 'directed' i.e. directed ordering of objects and morphisms only go forward in the sequence.

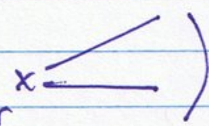
Define $\hom(\Delta_i, \Delta_j) = \begin{cases} 0 & \text{if } i>j \quad ① \\ \mathbb{K}\cdot e_i & \text{if } i=j \quad ② \\ CF(V_i, V_j) & \text{if } i<j \quad ③ \end{cases}$

① $\underline{i>j}$



CASE II situation, γ_i, γ_j
γ_j is pushed up so don't intersect.

② $\underline{i=j}$

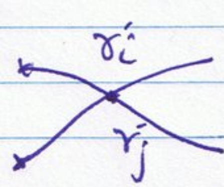


intersection point

upstairs:

1 thimble, pushing it up what remains is ~~the~~ one intersection point
The 2 paths & the 2 thimbles intersect transversally at the c.p.
e_i - for identity of this object.
$\mathbb{K}$ - the field.

③ $\underline{i<j}$



CASE I situation.
Pick up intersection b/w vanishing cycles inside that fiber.

<u>Note</u>: $CF(V_i, V_j)$ is combinatorial, involves data in the fibers.

- μ^k in $\mathcal{F}^{\rightarrow}(\{\gamma_i\})$:

  $\hom(\Delta_{i_{k-1}}, \Delta_{i_k}) \otimes \cdots \otimes \hom(\Delta_{i_0}, \Delta_{i_1}) \to \hom(\Delta_{i_0}, \Delta_{i_k})[\;]$

  * zero unless $i_0 \le i_1 \le \dots \le i_k$ (very clear, otherwise there are no morphisms to compare)

  * e_i strict unit: $\mu^2(x, e_i) = \mu^2(e_i, x) = x$
    $\mu^{k\neq 2}(\dots, e_i, \dots) = 0$

General case: * $i_0 < \dots < i_k$: μ^k in $\mathcal{F}(M)$

The philosophy for defining $\mathcal{F}(\pi)$:

Take $\mathcal{F}(E)$ & take this derived category $\vec{\mathcal{F}}(\{\gamma_i\})$ put them together into a single thing.
"Make the Lagrangians & thimbles play with each other." D.A.

The actual defn of $\mathcal{F}(\pi)$ involves a double cover (trick).

Let $\tilde{E} = \{(x,y) \in E \times \mathbb{C} \mid y^2 = \pi(x) - 1\}$

$\tilde{E}$ is clearly a double cover of E. (For ea. value of x there are 2 values of y.) Branched at $\pi^{-1}(1) \equiv M$

1 stands for basepoint. Secretly still thinking of unit disc with 1.

$$\left.\begin{array}{l} \tilde{E} \\ \big\downarrow \tilde{\pi} =: \text{projection using } y \\ \mathbb{C} \end{array}\right\} \text{is a Lefschetz fibration}$$

$\mathbb{C}$: later can truncate to get D^2.

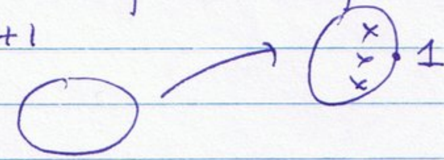
Why is it a Lefschetz fibration?

$\tilde{\pi}^{-1}(y) = (x, y) = (\pi^{-1}(y^2+1), y)$.
(x,y) s.t. $y^2 = \pi(x) - 1$

The fiber in E above y is $\pi^{-1}(y^2+1)$.

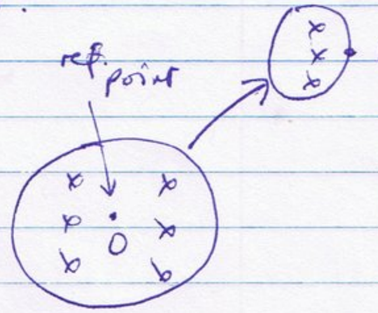
All we're doing is taking the double cover of the disc.

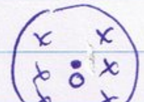
$y \mapsto y^2+1$



looking at the fiber prdct.

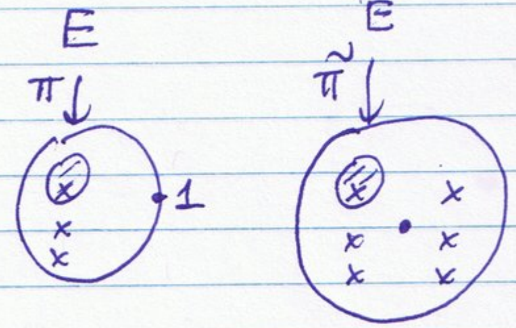
This is a Lef. fib. with same fiber as old one but twice as many singular fibers & they just got duplicated about the reference pt.



- $\tilde{E} \xrightarrow{\tilde{\pi}} \mathbb{C}$ is a Lefschetz fibration with $\tilde{\pi}^{-1}(0) = M$,
  critical values $= \{y \mid y^2+1 \in \text{critical values}(\pi)\}$.
  It carries a $\mathbb{Z}/2$ - action ($y \mapsto -y$) switching the 2 halves of .

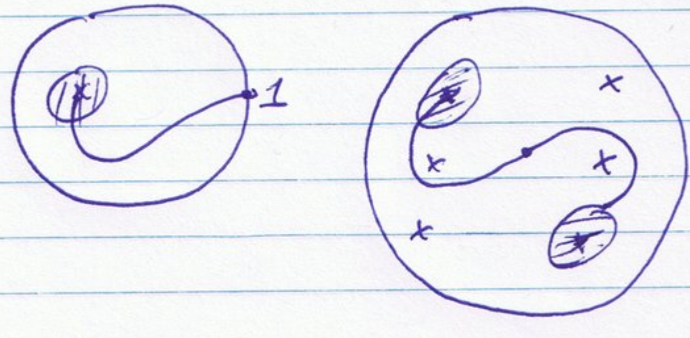
2 kinds of $\mathbb{Z}/2$-equivariant Lagrangians

- type (U): compact $L \subset \mathrm{int}(E) \rightsquigarrow \tilde{L}$ = lift of L to $\tilde{E}$
  $= \tilde{L}_+ \cup \tilde{L}_-$
  (disjoint, 2 copies of L)



- type (B): $\tilde{\Delta}$ = lift of thimble Δ to E.
  (smooth Lag.-sphere in $\tilde{E}$)

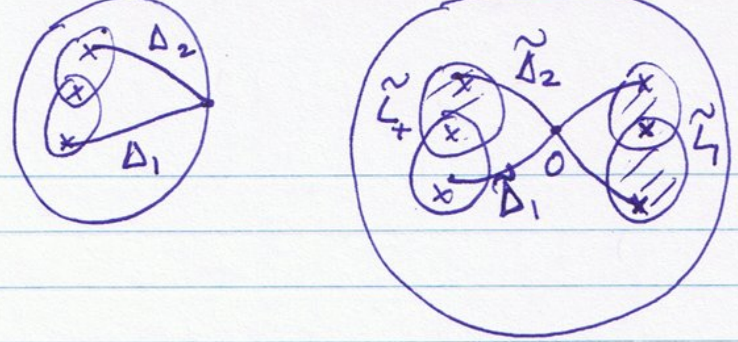
  consisting of 2 copies of the thimble left & right.



Defn: $\mathcal{F}(\pi)$ =: subcategory of $\mathcal{F}(\tilde{E})$ with:

- objects := type (U) & type (B) Lagrangians ← double lifts of Lagrangians in E & of thimbles.
- morphisms = $\mathbb{Z}/2$-invariant part of $\hom_{\mathcal{F}(\tilde{E})} = CF_{\tilde{E}}$

needs char$(\mathbb{K}) \neq 2$.



<u>Miracle</u>: $\mathbb{Z}/2$-action on $CF(\tilde{\Delta}_1, \tilde{\Delta}_2)$ is identity.
" $CF(\tilde{\Delta}_2, \tilde{\Delta}_1)$ is $-$identity.

Now that we've defined this category, how does it relate to other things?

$\vec{\mathcal{F}}(\{\gamma_1, \dots, \gamma_n\})$ & $\mathcal{F}(E)$ are 2 full & faithful subcategories of $\mathcal{F}(\pi)$.

$$\vec{\mathcal{F}}(\{\gamma_1, \dots, \gamma_n\}) \hookrightarrow \mathcal{F}(\pi)$$
$$\Delta_i \longmapsto \tilde{\Delta}_i$$

$$\mathcal{F}(E) \hookrightarrow \mathcal{F}(\pi)$$
$$L \longmapsto \tilde{L}$$

<u>Punchline</u>: These objects generate the whole thing. We'll see how this goes.

Observe $\tilde{E}$ is itself a fiber of a Lefschetz fibration.

$$\tilde{\mathcal{E}} = \{(x, y, w) \in E \times \mathbb{C} \mid y^2 = \pi(x) - w\} \simeq \underset{x}{E} \times \underset{y}{\mathbb{C}}$$

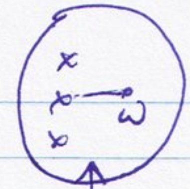
$$\downarrow w = \pi(x) - y^2$$

$\mathbb{C}$ critical points $= (\mathrm{crit}\,\pi) \times \{0\}$
critical values $=$ crit values (π).

Fiber looks like $\tilde{E}_w = \{(x, y) \in E \times \mathbb{C} \mid y^2 = \pi(x) - w\}$

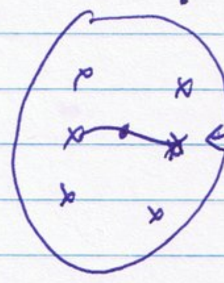
What does it look like ?

Old Lefschetz Fib.



As ω approaches a critical value this double path gets shorter

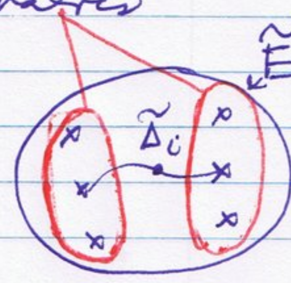
send ω to origin & then do a $\sqrt{\ }$ & call it y.

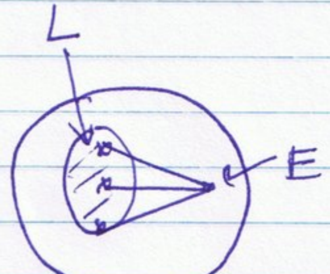


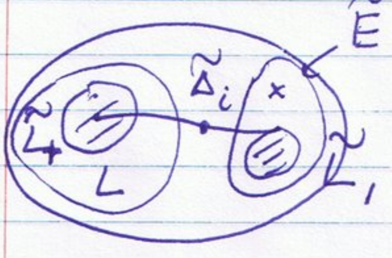
<u>Claim</u>: Fiber is $\simeq \tilde{E}$
critical values = critical values(π)

- for vanishing path γ_i
  vanishing cycle $\tilde{\Delta}_i \subset \tilde{E}$

- $\tau_{\tilde{\Delta}_1} \cdots \tau_{\tilde{\Delta}_m} \simeq$ total monodromy
  switches the 2 halves







<u>Lemma</u>:

$$\tau_{\tilde{\Delta}_1} \cdots \tau_{\tilde{\Delta}_m}(\tilde{L}_+) \cong \tilde{L}_-[1] \text{ in } \mathcal{F}(\tilde{E}).$$

Recall: $T_{\tilde{\Delta}_1} \cdots T_{\tilde{\Delta}_m}(\tilde{L}_+) \cong$
twisted complex

$$\begin{tikzcd}
CF(\tilde{\Delta}_m, \tilde{L}_+) \otimes \tilde{\Delta}_m \arrow[r] & \tilde{L}_+ \\
CF(\tilde{\Delta}_m, \tilde{L}_+) \otimes CF(\tilde{\Delta}_{m-1}, \tilde{\Delta}_m) \otimes \tilde{\Delta}_{m-1} \arrow[r] \arrow[u, leftrightarrow] & CF(\tilde{\Delta}_{m-1}, \tilde{L}_+) \otimes \tilde{\Delta}_{m-1} \arrow[u, "\simeq"]
\end{tikzcd}$$

i.e. $\left(\text{mapping cone of } \boxed{\begin{array}{l}\text{twisted complex} \\ \text{of } \tilde{\Delta}_1 \cdots \tilde{\Delta}_m\end{array}} \to \tilde{L}_+ \right) \simeq \tilde{L}_-[1]$

Consequence
$$\begin{tikzcd}
\boxed{} \arrow[r] & \tilde{L}_+ \arrow[dl, "{0}" ', "{[1]}"] \\
\tilde{L}_- \arrow[u] &
\end{tikzcd}$$
exact triangle in $T_w \mathcal{F}(\tilde{E})$

Since $hom(\tilde{L}_+, L_-[1]) = 0$ $\quad (\tilde{L}_+ \cap \tilde{L}_- = \phi)$

get $\boxed{} \underset{\substack{\text{quasi-isom.} \\ \text{in } T_w \mathcal{F}(\tilde{E})}}{\simeq} \tilde{L}_+ \oplus \tilde{L}_1 \simeq \tilde{L}$.

Pass to $\mathbb{Z}/2$-invariant part $\Rightarrow$

in $T_w \mathcal{F}(\pi)$, $\boxed{} \simeq \tilde{L}$.

i.e. $\tilde{L}$ is generated by $\tilde{\Delta}_1 \cdots \tilde{\Delta}_m$.

<u>Cor 1</u>: $\mathcal{F}(\pi)$ is generated by $\tilde{\Delta}_1 \ldots \tilde{\Delta}_m$.

[We showed how the compact Lagrangians are expressed in terms of $\tilde{\Delta}_1, \ldots, \tilde{\Delta}_m$.]

<u>Cor 2</u>: $\boxed{\underbrace{\mathcal{F}(E)}_{\substack{\text{reduced} \\ \text{understanding} \\ \text{Fuk Cat of this}}} \hookrightarrow T_w \mathcal{F}(\pi) \simeq \underbrace{T_w \mathcal{F}^{\to}(\{\gamma_i\})}_{\substack{\text{in terms of understanding} \\ \text{these}}}}$